Chebyshev Polynomials of the First Kind

Start with formulas for the cosine of the sum and difference of angles:

$$\cos[(n+1)\theta] = \cos(n\theta + \theta) = \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta$$

$$\cos[(n-1)\theta] = \cos(n\theta - \theta) = \cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta$$

$$\cos[(n+1)\theta] + \cos[(n-1)\theta] = 2\cos(n\theta)\cos\theta$$

$$\cos[(n+1)\theta] = 2\cos\theta\cos(n\theta) - \cos[(n-1)\theta] \longleftarrow \text{ a recursive formula for } \cos[(n+1)\theta]$$

Convert to a polynomial by defining: $x = \cos\theta\,,\ T_n(x) = \cos(n\theta)$

$$T_0(x) = \cos(0\cdot\theta) = \cos 0 = 1$$

$$T_1(x) = \cos(1\cdot\theta) = \cos\theta = x$$

$$\dots$$

$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x)$$

Some Chebyshev Polynomials of the First Kind:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$

$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$

$$T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$$

$$T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$$

Some things to notice:

1) What do you notice about all of the coefficients?

2) What do you notice about the signs of consecutive terms?

3) What do you notice about the exponents in $T_n(x)$ when n is even?

4) What do you notice about the exponents in $T_n(x)$ when n is odd?

5) What do you notice about the constant term?

6) What do you notice about the coefficients of the lead terms?

7) What do you notice about the coefficients of the x-terms?

8) What do you notice about the coefficients of the x^2-terms?

9) What do you notice about the sum of the coefficients in any $T_n(x)$?

The graph to the right shows a plot of $T_0(x)$ to $T_5(x)$ on the interval $[-1, 1]$. The roots of $T_n(x)$ are called Chebyshev nodes and can be used to interpolate between known points in a set of data.